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Electron states in a magnetic field and random impurity potential: use of the theory of entire functions

Y Avishai†, R M Redheffer‡ and Y B Band§

† Department of Physics, Ben Gurion University of the Negev, Beer Sheva, Israel
‡ Department of Mathematics, UCLA, Los Angeles, CA 90024, USA
§ Department of Chemistry, Ben Gurion University of the Negev, Beer Sheva, Israel, and Institute for Theoretical Atomic and Molecular Physics, Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, MA 02138, USA

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Abstract. We determine the wavefunction of an electron in the presence of a transverse magnetic field and randomly located delta function impurities in the x–y plane within the subspace of the lowest Landau level. The wavefunction contains as a factor an entire function of $z=x+iy$ which vanishes at all impurity sites. The question of whether such states are extended is related to the rate of growth of an entire function in terms of the distribution of its zeros. For a homogeneous distribution of impurities the corresponding Weierstrass product is of order 2 and of finite type. This rate of growth can be exactly compensated by the Gaussian term due to the presence of the magnetic field such that there is a critical field beyond which extended states may exist. If the impurities are located on the sites of a square lattice, the extended states are given in closed form in terms of the Weierstrass σ function.

In this paper we study electron states in the two-dimensional x–y plane subject to a strong magnetic field and a disordered potential given by a sum of delta function terms (referred to as impurities). More specifically, we address the question of whether extended states exist whose energy equals the original Landau energy without the disordered potential. The relation of this question to the interpretation of the integer quantum Hall effect is well known [1]. The occurrence of extended states on the Landau levels and localized states between levels results in the step structure of σ_{xy} and the spike structure of σ_{xx} as the magnetic field is varied. The proof of the extended and localized nature of the wavefunction is in general indirect. Here we adopt a direct approach and formulate some general statements about the nature of the eigenstates.

The central concept of the present solution is the use of ideas from the theory of entire functions. As it turns out, the wavefunction contains as a factor an entire function which vanishes on the sites of the impurities. This approach leads to the conclusion that the question of whether solutions at the Landau level energies exist is related to the growth rate of an entire function (i.e. its order and its type) whose distribution of zeros is given. If the distribution of impurities is uniform, the relevant order of the pertinent Weierstrass product is 2 and the type is finite. It is precisely this fortunate geometrical property which, when combined with the physical properties of an electron in a magnetic field, assures the existence of solutions to the original problem.

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The Hamiltonian in the Landau gauge $A_x = By$, with magnetic length $L = \sqrt{\hbar c/|eB|}$ is

$$H = \left\{ -\frac{\partial^2}{\partial y^2} + \left[\mathrm{i}\frac{\partial}{\partial x} + \frac{y}{L^2} \right]^2 \right\} + \sum_{m=1}^{\infty} v_m \delta(x - x_m)\delta(y - y_m) = H_0 + V \tag{1}$$

where the potential strengths, v_m, are random numbers and (x_m, y_m) are the random locations of impurities arranged in order of increasing distance from the origin. It is useful to denote the complex position of the impurities as $z_m = x_m + \mathrm{i}y_m$. The distribution of the impurities in the two-dimensional plane is of crucial importance in what follows. Loosely speaking, we require a finite density, some sort of uniformity and a degree of symmetry. By a finite density we mean that the number of impurities in a circle of radius r, $N(r)$, divided by the area of this circle tends to a constant n as $r \to \infty$,

$$n = \lim_{r\to\infty} \frac{N(r)}{\pi r^2} = \frac{1}{d^2}. \tag{2}$$

By uniformity we mean the following: Let $\{\Omega_m\}$ be the complex point set in a two-dimensional square lattice of length d arranged in order of increasing distance from the origin (two points with the same distance are arranged according to their polar angle). It is then required that the distance between an impurity z_m and its corresponding lattice point Ω_m will not be large in the sense that

$$\sum_{m=1}^{\infty}{}' \left| \frac{z_m - \Omega_m}{\Omega_m^3} \right| < \infty \tag{3}$$

where the prime denotes that the origin is excluded from the sum. Finally, by symmetry we mean that the function

$$Q(r) = \sum_{|z_m|<r} \frac{1}{z_m^2} \tag{4}$$

is a bounded function of r. It is easily seen that (2), (3) and (4) are three independent conditions.

The basic question addressed here concerns the asymptotic behaviour of the solutions of the Schrödinger equation

$$H\Psi(x, y) = E\Psi(x, y). \tag{5}$$

The eigenfunctions and eigenvalues of H_0 are the Landau functions $L_{kn}(x, y)$ $(-\infty < k < \infty)$ and Landau energies $E_n = (2n+1)/L^2$. For each n, the functions $L_{kn}(x, y)$ form a subspace P_n (which also denotes the projection operator on that space). If the magnetic field is sufficiently strong, a projection onto a single Landau level constitutes a reasonable approximation. The problem then becomes essentially one-dimensional and the difficulties related to the presence of delta function interactions in $D > 1$ are eliminated. We then focus our attention on the solution $\psi(x, y)$ of the Schrödinger equation for the projected Hamiltonian $P_0 H P_0$. The kinetic energy operator is replaced by the constant $P_0 H_0 P_0 = E_0$ while the potential energy $P_0 V P_0$ now becomes a non-local operator whose configuration space representation is $\langle xy|P_0 V P_0|x'y'\rangle$:

$$\int \langle xy|P_0 V P_0|x'y'\rangle \psi(x', y')\, \mathrm{d}x'\, \mathrm{d}y' = (E - E_0)\psi(x, y). \tag{6}$$

The precise form of the kernel $\langle xy|P_0 V P_0|x'y'\rangle$ is not given here since instead of (6) we study the analogous equation in (pseudo-) momentum space where the form of

the integral kernel is given explicitly in (10) below. By performing the projection, the problem becomes essentially one-dimensional. The transformation of wavefunctions and operators to momentum space is carried out in terms of the $n = 0$ Landau functions†

$$L_{k0}(x, y) \equiv L_k(x, y) = \frac{1}{\sqrt{2\pi}} (L^2\pi)^{-1/4} \exp\left[ikx - \frac{1}{2}\left(\frac{y}{L} - kL\right)^2\right]. \tag{7}$$

Thus, if $\psi(x, y)$ is the solution of (6), then

$$f(k) = \int L_k(x, y)\psi(x, y)\,dx\,dy \tag{8}$$

is a solution of

$$\int v(k, k')f(k')\,dk' = (E - E_0)f(k). \tag{9}$$

The momentum space representation of the potential is given explicity by

$$\begin{aligned} v(k, k') &= \int L_k(x, y)\langle xy|P_0 V P_0|x'y'\rangle L_{k'}^*(x', y')\,dx\,dy\,dx'\,dy' \\ &= \sum_m v_m g_m(k) g_m^*(k'). \end{aligned} \tag{10}$$

This is an infinite sum (over impurities) of separable terms with 'form factors' $g_m(k)$ given by

$$g_m(k) = \exp\left[ikx_m - \frac{1}{2}\left(\frac{y_m}{L} - kL\right)^2\right]. \tag{11}$$

If the solution $f(k)$ of the integral equation (9) is known, then the wavefunction in configuration space is recovered by applying the inverse Fourier transform. Hence,

$$\begin{aligned} \psi(x, y) &= \int L_k^*(x, y)f(k)\,dk = \int \exp\left[-ikx - \frac{1}{2}\left(\frac{y}{L} - kL\right)^2\right] f(k)\,dk \\ &= \exp\left(-\frac{y^2}{2L^2}\right) \int \exp[-ikz - \tfrac{1}{2}(kL)^2] f(k)\,dk \end{aligned} \tag{12}$$

where in the least term we anticipate the central role of the complex variable $z = x + iy$.

One may wonder why we use both configuration and momentum space representations. The reason is that the solution of the Schrödinger equation is simpler in momentum space where a one-dimensional integral equation (9) results. Yet, test of localization or extendedness is simpler in configuration space. Note that $v(k, k')$ is non-local (a local integral operator in momentum space depends on k and k' only through $k - k'$). Thus, it is reasonable to expect the existence of extended states even though a one-dimensional problem with a random potential is involved since Anderson localization in 1D pertains to local potentials.

† Throughout this paper, configuration space variables $x, y, z, \ldots$ have dimension of length while momentum space variables $k, q, \ldots$ have dimension of inverse length. Functions in configuration space have dimensions of inverse square root of length, while functions in momentum space have dimensions of length. The entire function $F(z)$ (see Eq. (13)) is dimensionless. All integrals are from $-\infty$ to $+\infty$.

The remainder of this paper concerns the solution of equation (9) for $E = E_0$, namely

$$\sum_{m=1}^{\infty} v_m g_m(k) \exp\left(-\frac{y_m^2}{2L^2}\right) \int \exp[-ik'z_m - \tfrac{1}{2}(k'L)^2] f(k')\, dk' = 0. \tag{13}$$

If we find a non-trivial function $f(k)$ such that

$$\int \exp[-ikz_m - \tfrac{1}{2}(kL)^2] f(k)\, dk = 0 \tag{14}$$

for all m, this function obviously satisfies (13). Conversely, if (13) holds for all real k and for a random set of impurity strengths $\{v_m\}$ with each $v_m \neq 0$, we would expect the validity of (14). In this sense (13) and (14) are equivalent.

We are thus led to an orthogonality problem of a complex function $f(k)$ of a real variable k, to the set of exponentials $\{e^{-ikz_m}\}$ multiplied with Gaussian weight $e^{-\frac{1}{2}(kL)^2}$. The following question arises: Under what conditions is the set of functions $\{e^{-ikz_m}\}$ on the sequence $\{z_m\}$ *multiplied with Gaussian weight* $e^{-\frac{1}{2}(kL)^2}$ *(over-) complete?* For a finite interval ($0 \leq k \leq K$), this problem has a long history†. In this case the Gaussian weight is not needed to assure the convergence of the integrals in (13) and (14). For the infinite interval, the Gaussian weight is necessary, and the problem is characterized by two length scales: L and d. Notice that the existence of a solution $f(k)$ in the sense described above is dependent neither on the magnitudes nor on the signs of the strengths $\{v_m\}$ as long as $v_m \neq 0$. This striking result [3] is true only when the solution of the original problem is restricted to the Landau energies. On the other hand, the location of impurities and their density is of crucial importance.

As noted [2], orthogonality to a set of exponential functions is related to the theory of entire functions. Suppose we have a solution $f(k)$ in hand. We can define a function $F(z)$ of a complex variable z:

$$F(z) = \int \exp[-ikz - \tfrac{1}{2}(kL)^2] f(k)\, dk. \tag{15}$$

We limit ourselves to a class of functions $f(k)$ for which the integral is uniformly convergent in each disk $|z| \leq R$. Then (i) $F(z)$ is an entire function of z. (ii) Equation (14), implies

$$F(z_m) = \int \exp[-ikz_m - \tfrac{1}{2}(kL)^2] f(k)\, dk = 0 \tag{16}$$

i.e. $F(z)$ vanishes on the set of impurities. (iii) If we now limit equation (15) to $z = x$, we see that $F(x)$ is the Fourier transform of $e^{-\frac{1}{2}(kL)^2} f(k)$. Hence $1/2\pi \int_{-\infty}^{\infty} e^{ikx} F(x)\, dx$ exists for all k, except possibly for a countable number of points, (we do not exclude the possibility that $f(k)$ includes a sum of delta functions), and $f(k)$ can be constructed from $F(x)$ by the inverse Fourier transform of (15) when restricted to $z = x$.

Our strategy is to start from an entire function $F(z)$ such that $F(z_m) = 0$, and to construct $f(k)$ from the given $F(z)$. This manipulation can be performed by noticing that $F(z)$ is determined by its restriction to the real axis $z = x$.

† For references related to the pertinent orthogonality problem and its relation to the theory of entire functions, see [2].

We now ask the question, what are the properties of the configuration space wavefunction $\psi(x, y)$ whose momentum space Fourier transform $f(k)$ is given by

$$f(k) = e^{\frac{1}{2}(kL)^2} \int_{-\infty}^{\infty} e^{ikx} F(x)\, dx \tag{17}$$

where $F(x)$ is the restriction to the real axis of an entire function $F(z)$ satisfying conditions (i)–(ii)? Inspecting equation (12), one finds that in fact $\psi(x, y)$ is related directly to $F(z)$ through

$$\psi(x, y) = \exp\left(-\frac{y^2}{2L^2}\right) F(z). \tag{18}$$

In other words, the solution of $P_0 H P_0 \psi = E_0 P_0 \psi$ is a Gaussian in y times a function $F(z)$ with the properties (i)–(iii). Thus, in order to investigate the behaviour of the wavefunction $\psi(x, y)$ we construct the entire function $F(z)$. The properties of the wavefunction in configuration space are then determined by the rate of growth of an entire function $F(z)$ whose distribution of zeros is given. The following statements pertaining to the theory of entire functions are based on results from Boas [4].

Our first task is to study the Weierstrass product for the sequence of impurities. For some $\beta > 0$ we examine the series

$$C(\beta) = \sum_{m=1}^{\infty} \frac{1}{|z_m|^{\beta}} \tag{19}$$

and remember that for our impurity distribution $|z_m| \approx d\sqrt{m}$; hence the series converges for any $\beta > 2$ but diverges for $\beta = 2$. According to Boas [5, 2.5.2] the convergence exponent of the sequence $\{z_m\}$ is $\rho = 2$, and according to [5, 2.5.4] the genus of the sequence is $p = 2$. By [5, 2.6.5] the Weierstrass product of the impurity sequence (assuming for simplicity $z_m \neq 0$),

$$W(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{z_m}\right) \exp\left(\frac{z}{z_m} + \frac{z^2}{2z_m^2}\right) \tag{20}$$

is an entire function of order $\rho = 2$ [5, 2.1.1]). Evidently, the Weierstrass product (20) must be included as a factor in the entire function $F(z)$. To inspect the type of the Weierstrass product [5, 2.1.3] we use the Lindelöf theorem [5, 2.10.1]: Since $W(z)$ has an *integer* order ($\rho = 2$ in our case), then the symmetry requirement implies that $W(z)$ has a finite type τ.

Having assured the vanishing of $F(z)$ on the impurities, we still have some freedom in constructing $F(z)$ by multiplying the Weierstrass product $W(z)$ with an entire function $h(z)$ of order at most 2 and finite type τ' which has no zeros in the whole complex plane. Thus we may write

$$\psi(x, y) = \exp\left(-\frac{y^2}{2L^2}\right) F(z) = \exp\left(-\frac{y^2}{2L^2}\right) W(z) h(z) \tag{21}$$

where $h(z)$ is an entire function of order $\leqslant 2$ and type $\tau' < \infty$.

Can one use the freedom in the choice of $h(z)$ in order to control the behaviour of $\psi(x, y)$ as x and y tend to infinity? We are tempted to conjecture that in some cases, the answer to this question is positive, as it is evident in the particular situation where the impurities are located on a 2D square lattice of length a (for simplicity set $a = 1$), namely $z_m = \Omega_m$. The potential is then

$$V(x, y) = \sum_m v_m \delta(r - \Omega_m) \tag{22}$$

and the disorder is implemented through the strengths v_m and not through the positions. The Weierstrass product is then

$$W(z)=\sigma(z)=z\prod_{\Omega_m\neq 0}\left(1-\frac{z}{\Omega_m}\right)\exp\left(\frac{z}{\Omega_m}+\frac{z^2}{2\Omega_m^2}\right). \tag{23}$$

It has the representation [5]

$$\sigma(z)=e^{\eta z^2}S(z) \tag{24}$$

where, in the notation of [5] $\eta=\zeta(1/2)$ here ζ is the Weierstrass zeta function, and $S(z)$ is bounded and oscillatory on the real axis. We choose $h(z)=\exp(-\eta z^2)$ so that our entire function is simply $S(z)$, and the wavefunction is

$$\psi(x,y)=\exp\left(-\frac{y^2}{2L^2}\right)S(z) \tag{25}$$

which is bounded *and extended* on the real axis. It is also bounded on the imaginary axis if $S(z)$ blows up on the imaginary axis slower than the Gaussian. This is always achieved for small enough L (large enough B) since the type of $S(z)$ is finite. In fact, η is the type of $\sigma(z)$, and since $S(z)=e^{-\eta z^2}\sigma(z)$ it grows on the imaginary axis at most as $e^{2\eta y^2}$. Hence the condition $1/L^2>4\eta$ assures boundedness on the imaginary axis and leaves extendedness on the real axis unaffected. Thus, we have established the existence of a critical field above which states are extended. Curiously, $\eta=\pi/2$ and hence the condition $1/L^2=4\eta$ implies a flux unit through an elementary square. The value of the critical magnetic field required for delocalization of states on the lowest Landau level is then given by $B=hc/ea^2$. This point, and a generalization of the present result to all Landau levels will be discussed elsewhere [6].

We conclude with several remarks:

(1) If one moves the positions of any finite number of impurities $z_m\to w_m$, $m=1,2,\ldots,M$, then the solution is obtained from that of equation (21) upon multiplying it by

$$\prod_{m=1}^{M}\frac{z-w_m}{z-z_m}.$$

(2) This result shows that extendedness can be determined up to a power. If we remove (or add) a finite number of impurities z_m, $m=1,2,\ldots,M$, we must divide (or multiply) our previous solution by the factor

$$\prod_{m=1}^{M}(z-z_m)$$

which affects the behaviour of the solution at infinity up to a finite power M. The addition or removal of any finite number of impurities cannot of course change the essential physics.

(3) The extendedness property derived here is independent of gauge. However, if one chooses a symmetric gauge then the states are localized, but an infinite degeneracy associated with choice of origin exists. Extended states are formed in the symmetric gauge as appropriate linear combinations of the degenerate localized states.

(4) Linear combinations of states from a single Landau level (such as $\psi(x,y)$ in (12)) do not carry current. Current results once the system is attached to leads and edge states are taken into account [7].

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